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Noncommutative Kähler-like structures in quantization

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Abstract

The notion of C^*-algebra with polarization which could be considered as a quantum Kähler structure is introduced. The connection of these algebras with Kostant–Souriau geometric quantization is shown. The theory of polarized C^*-algebra is investigated by the use of the coherent states method.

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1. Introduction

There are complementary methods of mathematical description of quantum physical systems. These are the algebraic methods based on the theory of C^*-algebras, see [1], and the geometric methods that find an elegant presentation as Kostant–Souriau quantization and $*$-product quantization, see [2,3].

In our approach we make an effort to unify these two methods by use of the notion of the coherent states map [4]. By the coherent states map $\mathcal{K}$ we mean a symplectic map of the classical phase space M into quantum phase space, i.e. complex projective Hilbert space $\mathbb{CP}(\mathcal{M})$. It appears that using the coherent states map one can unify, in some sense, the classical and quantum descriptions of the physical system considered. Furthermore the system is defined by $\mathcal{K}$ and a Hamiltonian satisfying some condition of consistency with $\mathcal{K}$, see [5,4,6].

Bearing in mind the above fact we introduce in Section 4 the notion of a polarized algebra of observables $\mathcal{A}$, which is unequivocally determined by the coherent states map $\mathcal{K}$. In the case when $\mathcal{K}$ is the Gaussian coherent states map of the linear phase space $\mathbb{R}^{2N}$ into $\mathbb{CP}(\mathcal{M})$, the algebra $\mathcal{A}$ is the Heisenberg–Weyl algebra. So the C^*-algebra $\mathcal{A}$ is a natural generalization of the latter to the case of a general phase space M. We prove some important properties of $\mathcal{A}$ and explain the relation of the structure of $\mathcal{A}$ to structures such as the pre-quantum line bundle and polarization which play a crucial role in Kostant–Souriau quantization.

As a result we can distinguish additional structures on the C^*-algebra $\mathcal{A}$ that are responsible for the Kähler structure of classical phase space M. This structure denotes the existence of a commutative Banach subalgebra $\overline{\mathcal{P}}$ of $\mathcal{A}$ which

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has a physical interpretation as the algebra of annihilation operators. Its classical counterpart is the Kähler polarization in the sense of Kostant–Souriau quantization. So, it is natural to understand $\overline{\mathcal{P}}$ as the quantum polarization and call $(\mathcal{A}, \overline{\mathcal{P}})$ a polarized C^*-algebra or quantum Kähler manifold.

In Section 5 we introduce the notion of an abstract coherent state on $(\mathcal{A}, \overline{\mathcal{P}})$. The coherent states in this sense generalize the notion of a vacuum to the case of a general phase space. On the other hand using the coherent states one can study the algebra $\mathcal{A}$ by reducing many problems to the investigation of its polarization $\overline{\mathcal{P}}$, which is handier because of commutativity.

Also in Section 5 we show the fundamental properties of coherent states: on $\mathcal{A}$ they can be considered as classical states of some classical phase space that is the subspace of the space of multiplicative functionals on the polarization $\overline{\mathcal{P}}$. Additionally, when we apply the GNS construction to the coherent states we obtain a Hilbert space which is a generalization of the Hardy space [7,8], which is exactly obtained when $M = \mathbb{D}$ is a unit disc in $\mathbb{C}$ and $\mathcal{A}$ is the Toeplitz algebra [7].

In Section 6 we show how to reconstruct from $(\mathcal{A}, \overline{\mathcal{P}})$ the classical phase space M and the coherent states map $\mathcal{K}$. This gives rise to the method of reconstruction of the classical mechanics picture for the quantum one.

Finally we would like to note that the results presented in the paper are in fact a step in the direction of a theory of physical systems which unifies their classical and quantum descriptions.

2. Coherent state map and polarization

Let us fix a map $\mathcal{K} : M \to \mathbb{CP}(\mathcal{M})$ of the manifold M into complex projective Hilbert space $\mathbb{CP}(\mathcal{M})$. Assume that $\mathcal{K}$ is of the same smoothness class as M and that the image $\mathcal{K}(M)$ is linearly dense in the Hilbert space $\mathbb{CP}(\mathcal{M})$. This space will be always assumed to be separable. For the description of the map $\mathcal{K}$ as well as all related objects one needs to fix a local trivialization

$$K_\alpha : \Omega_\alpha \longrightarrow \mathcal{M} \setminus \{0\} \tag{1}$$

where $[K_\alpha(m)] = \mathcal{K}(m)$ for $m \in \Omega_\alpha$ and

$$K_\alpha(m) = g_{\alpha\beta}(m) K_\beta(m) \tag{2}$$

for $m \in \Omega_\alpha \cap \Omega_\beta$ of the map $\mathcal{K}$. The open sets Ω_α, $\alpha \in J$, cover the manifold M. The transition functions

$$g_{\alpha\beta} : \Omega_\alpha \cap \Omega_\beta \longrightarrow \mathbb{C} \setminus \{0\}$$

form a smooth cocycle:

$$g_{\alpha\gamma}(m) g_{\gamma\beta}(m) = g_{\alpha\beta}(m), \quad m \in \Omega_\alpha \cap \Omega_\beta \cap \Omega_\gamma \tag{3}$$

on the manifold M.

Let $\mathbb{L} \to M$ denote the complex line bundle over M which is the pull-back $\mathbb{L} := \mathcal{K}^*\mathbb{E}$ of the universal bundle:

$$\mathbb{E} = \{(v, l) \in \mathcal{M} \times \mathbb{CP}(\mathcal{M}) : v \in l\} \xrightarrow{pr_2} \mathbb{CP}(\mathcal{M}). \tag{4}$$

From the definition above it follows that $\mathbb{E} \to \mathbb{CP}(\mathcal{M})$ is a holomorphic line bundle equipped with Hermitian metric H_{FS} which is defined by the scalar product $\langle\cdot|\cdot\rangle$ in the Hilbert space $\mathcal{M}$. These two structures uniquely determine the compatible connection ∇_{FS} (see e.g. [9]). Since $i\,\mathrm{curv}(\nabla_{FS}) =: \omega_{FS}$ is the Fubini–Study form it is natural to call

$$(\mathbb{E} \to \mathbb{CP}(\mathcal{M}), \nabla_{FS}, H_{FS}, \omega_{FS}) \tag{5}$$

the Fubini–Study pre-quantum bundle. According to [10] it is the universal object of the category of pre-quantum bundles in the sense of Kostant [2,3]. The above means that for any pre-quantum bundle

$$(\mathbb{L} \to M, \nabla, H, \omega) \tag{6}$$

there exists a map $\mathcal{K} : M \to \mathbb{CP}(\mathcal{M})$ such that (6) is given as the pull-back of the Fubini–Study bundle:

$$\mathbb{L} = \mathcal{K}^*\mathbb{E}, \qquad \nabla = \mathcal{K}^*\nabla_{FS}, \qquad H = \mathcal{K}^* H_{FS}, \qquad \omega = \mathcal{K}^*\omega_{FS}. \tag{7}$$

Let us recall that objects presented in (6) satisfy appropriate consistency conditions and the condition $[\omega] \in H^2(M, \mathbb{Z})$.

In the terms of trivialization (1) the basic sections $s_\alpha : \Omega_\alpha \to \mathbb{L}$ are given by

$$s_\alpha(m) := (m, K_\alpha(m)), \quad m \in \Omega_\alpha \tag{8}$$

and the related potentials of the Hermitian metric H are

$$H(s_\alpha, s_\alpha) = \langle K_\alpha | K_\alpha \rangle, \tag{9}$$

while the connection form

$$\nabla s_\alpha =: \Theta_\alpha \otimes s_\alpha \tag{10}$$

can be written as

$$\Theta_\alpha = \frac{\langle K_\alpha | dK_\alpha \rangle}{\langle K_\alpha | K_\alpha \rangle}. \tag{11}$$

For the sake of completeness we give the expression for the curvature 2-form

$$\omega = i \operatorname{curv} \nabla = i d \frac{\langle K_\alpha | dK_\alpha \rangle}{\langle K_\alpha | K_\alpha \rangle}. \tag{12}$$

These formulae will be useful in the sequel.

Let $\overline{\mathbb{L}}^* \to M$ be the line bundle dual to the complex conjugate $\overline{\mathbb{L}} \to M$ of $\mathbb{L} \to M$. This bundle is also equipped with the metric structure $\overline{H}^*$ and the connection $\overline{\nabla}^*$. Their expressions in the gauge $\overline{s}_\alpha^* : \Omega_\alpha \to \overline{\mathbb{L}}^*$ are given by:

$$\overline{H}^*(\overline{s}_\alpha^*, \overline{s}_\alpha^*) = \frac{1}{\langle K_\alpha | K_\alpha \rangle} \tag{13}$$

$$\overline{\Theta}_\alpha^* = -\frac{\langle dK_\alpha | K_\alpha \rangle}{\langle K_\alpha | K_\alpha \rangle}, \tag{14}$$

where $\overline{\nabla}^* \overline{s}_\alpha^* = \overline{\Theta}_\alpha^* \otimes \overline{s}_\alpha^*$ on Ω_α.

Let us consider now the map:

$$I(v)(m) := \langle K_\alpha(m) | v \rangle \overline{s}_\alpha^*(m), \quad m \in \Omega_\alpha \tag{15}$$

of the Hilbert space $\mathcal{M}$ into the vector space $\Gamma^\infty(M, \overline{\mathbb{L}}^*)$ of the smooth sections of the bundle $\overline{\mathbb{L}}^*$. It is easy to check that the definition above does not depend on the choice of the gauge. The map I is a linear injection of $\mathcal{M}$ into $\Gamma^\infty(M, \overline{\mathbb{L}}^*)$. So the Hilbert space $\mathcal{M}$ can be identified with the vector subspace $I(M) \subset \Gamma^\infty(\mathcal{M}, \overline{\mathbb{L}}^*)$.

Let us consider the complex distribution $P \subset T^{\mathbb{C}}M$ spanned by smooth complex vector fields $X \in \Gamma^\infty(T^{\mathbb{C}}M)$ which annihilate the Hilbert space $I(\mathcal{M}) \subset \Gamma^\infty(M, \overline{\mathbb{L}}^*)$, i.e.

$$P := \bigsqcup_{m \in M} P_m, \tag{16}$$

where

$$P_m := \{X(m) : X \in \Gamma^\infty(T^{\mathbb{C}}M) \text{ and } \overline{\nabla}_X^* \psi = 0 \text{ for any } \psi \in I(\mathcal{M})\}. \tag{17}$$

To summarize the properties of P we formulate

Proposition 1. (i) *A necessary and sufficient condition for X to belong to $\Gamma^\infty(P)$ is*

$$\overline{X}(K_\alpha) = \Theta_\alpha(\overline{X}) K_\alpha. \tag{18}$$

(ii) *The distribution P is involutive and isotropic, i.e. for $X, Y \in \Gamma^\infty(P)$ one has*

$$[X, Y] \in \Gamma^\infty(P) \quad \textit{and} \quad \omega(X, Y) = 0. \tag{19}$$

(iii) *If* $X \in \Gamma^\infty(P \cap \overline{P})$ *then*

$$X \llcorner \omega = 0. \tag{20}$$

(iv) *Positivity condition:*

$$i\omega(X, \overline{X}) \geqslant 0 \tag{21}$$

for all $X \in \Gamma^\infty(P)$.

Proof. (i) By the definition one has that $X \in \Gamma^\infty(P)$ iff $\overline{\nabla}^*_X I(v) = 0$ for any $v \in \mathcal{M}$. From (14) and (15) we get

$$\begin{aligned}\overline{\nabla}^*_X I(v) &= X \llcorner \left(\langle dK_\alpha | v\rangle - \frac{\langle dK_\alpha|K_\alpha\rangle}{\langle K_\alpha|K_\alpha\rangle}\langle K_\alpha|v\rangle \right) \otimes \overline{s}^*_\alpha \\ &= \left\langle \overline{X}(K_\alpha) \middle| \left(\mathbb{I} - \frac{|K_\alpha\rangle\langle K_\alpha|}{\langle K_\alpha|K_\alpha\rangle}\right) v \right\rangle \overline{s}^*_\alpha = \langle \overline{X}(K_\alpha) - \Theta_\alpha(\overline{X})K_\alpha|v\rangle \overline{s}^*_\alpha.\end{aligned}$$

Thus we have proven (18).

(ii) From

$$d\Theta_\alpha = d\frac{\langle K_\alpha|dK_\alpha\rangle}{\langle K_\alpha|K_\alpha\rangle} = \frac{\langle dK_\alpha \wedge |dK_\alpha\rangle}{\langle K_\alpha|K_\alpha\rangle} - \frac{\langle dK_\alpha|K_\alpha\rangle \wedge \langle K_\alpha|dK_\alpha\rangle}{(\langle K_\alpha|K_\alpha\rangle)^2} \tag{22}$$

and (18) we obtain

$$\begin{aligned}d\Theta_\alpha(X, Y) &= \frac{1}{2}\left(\frac{\langle \overline{X}(K_\alpha)|Y(K_\alpha)\rangle - \langle \overline{Y}(K_\alpha)|X(K_\alpha)\rangle}{\langle K_\alpha|K_\alpha\rangle} - \right. \\ &\quad \left. - \frac{\langle \overline{X}(K_\alpha)|K_\alpha\rangle\langle K_\alpha|Y(K_\alpha)\rangle - \langle \overline{Y}(K_\alpha)|K_\alpha\rangle\langle K_\alpha|X(K_\alpha)\rangle}{(\langle K_\alpha|K_\alpha\rangle)^2}\right) \\ &= \frac{1}{2}(\overline{\Theta_\alpha(\overline{X})}\Theta_\alpha(Y) - \overline{\Theta_\alpha(\overline{Y})}\Theta_\alpha(X) - \overline{\Theta_\alpha(\overline{X})}\Theta_\alpha(Y) + \overline{\Theta_\alpha(\overline{Y})}\Theta_\alpha(X)) = 0\end{aligned} \tag{23}$$

for $X, Y \in \Gamma^\infty(P)$. Using the identity

$$d\Theta_\alpha(X, Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]} \tag{24}$$

and (23) we conclude that P is an involutive isotropic distribution.

(iii) Let $X \in \Gamma^\infty(P \cap \overline{P})$; then

$$\begin{aligned}\pounds_X\, \Theta_\alpha = \pounds_X \frac{\langle K_\alpha|dK_\alpha\rangle}{\langle K_\alpha|K_\alpha\rangle} &= -\frac{1}{\langle K_\alpha|K_\alpha\rangle}((\langle \overline{X}(K_\alpha)|K_\alpha\rangle \\ &\quad + \langle K_\alpha|X(K_\alpha)\rangle)\Theta_\alpha + \frac{1}{\langle K_\alpha|K_\alpha\rangle}(\langle \overline{X}(K_\alpha)|dK_\alpha\rangle + \langle K_\alpha|dX(K_\alpha)\rangle) \\ &= -(\overline{\Theta_\alpha(\overline{X})} + \Theta_\alpha(X))\Theta_\alpha + \overline{\Theta_\alpha(\overline{X})}\Theta_\alpha + \Theta_\alpha(X)\Theta_\alpha + d\,(\Theta_\alpha(X)) \\ &= d(\Theta_\alpha(X)) + X\llcorner d\Theta_\alpha - X\llcorner d\Theta_\alpha = \pounds_X\, \Theta_\alpha - X\llcorner d\Theta_\alpha.\end{aligned} \tag{25}$$

Hence one has (20).

(iv) For $X \in \Gamma^\infty(P)$ one can write

$$\begin{aligned}d\Theta_\alpha(X, \overline{X}) &= \frac{1}{2}\left(\overline{\Theta_\alpha(\overline{X})}\Theta_\alpha(\overline{X}) - \frac{\langle X(K_\alpha)|X(K_\alpha)\rangle}{\langle K_\alpha|K_\alpha\rangle} - \overline{\Theta_\alpha(\overline{X})}\Theta_\alpha(\overline{X}) + \frac{\langle X(K_\alpha)|K_\alpha\rangle\langle K_\alpha|X(K_\alpha)\rangle}{(\langle K_\alpha|K_\alpha\rangle)^2}\right) \\ &= -\frac{1}{2\|K_\alpha\|^2}(\|X(K_\alpha)\|^2\|K_\alpha\|^2 - |\langle K_\alpha|X(K_\alpha)\rangle|^2).\end{aligned} \tag{26}$$

Now from the Schwartz inequality one gets (23). □

Definition 2. Let $\mathcal{O}_\mathcal{K}$ denote the algebra of functions $\lambda \in C^\infty(M)$ such that $\lambda\psi \in I(\mathcal{M})$ if $\psi \in I(\mathcal{M})$.

In all further considerations we shall restrict ourselves to the cases of maps $\mathcal{K} : M \longrightarrow \mathbb{CP}(\mathcal{M})$ which satisfy the following conditions:

(a) *The curvature* 2*-form*

$$\omega = i\,\mathrm{curv}\nabla = \mathcal{K}^*\omega_{FS}$$

is non-degenerate, i.e. ω is symplectic.

(b) *The distribution P is maximal. i.e.*

$$\dim_{\mathbb{C}} P = \frac{1}{2}\dim M =: N. \tag{27}$$

(c) *For every $m \in M$ there exists an open neighborhood $\Omega \ni m$ and functions $\lambda_1, \ldots, \lambda_N \in \mathcal{O}_{\mathcal{K}}$ such that $d\lambda_1, \ldots, d\lambda_N$ are linearly independent on Ω.*

According to [4] the map $\mathcal{K}$ satisfying given conditions will be called a *coherent state map*. We present the following properties of the map $\mathcal{K}$.

Proposition 3. (i) *The manifold M is the Kähler manifold and $\mathcal{K} : M \longrightarrow \mathbb{CP}(\mathcal{M})$ is a Kähler immersion of M into $\mathbb{CP}(\mathcal{M})$.*

(ii) *The distribution P is the Kähler polarization of symplectic manifold (M, ω). Moreover P is spanned by the Hamiltonian vector fields X_λ generated by $\lambda \in \mathcal{O}_{\mathcal{K}}$.*

Proof. (i) The conditions (a) and (b) imply that P defines almost complex structure on M. The condition (c) guarantees its integrability. The property of M being Kähler follows from the fact that ω is symplectic and from positivity property (21) of Proposition 1. The immersion property of $\mathcal{K}$ follows from $\omega = \mathcal{K}^*\omega_{FS}$ and ω is symplectic.

(ii) Let us take $X \in \Gamma^\infty(P)$ and $\lambda \in \mathcal{O}_{\mathcal{K}}$. Then from

$$\overline{\nabla}^*_X \psi = 0 \quad \text{and} \quad \overline{\nabla}^*_X(\lambda\psi) = 0$$

for any $\psi \in I(\mathcal{M})$, it follows that $X(\lambda) = 0$. Let X_λ be the Hamiltonian vector field corresponding to λ

$$X_\lambda \llcorner \omega = d\lambda. \tag{28}$$

Then

$$\omega(X_\lambda, X) = d\lambda(X) = X(\lambda) = 0.$$

Since P is maximal isotropic one gets $X_\lambda \in \Gamma^\infty(P)$. The condition (c) implies now that P is spanned by X_λ where $\lambda \in \mathcal{O}_{\mathcal{K}}$. In this way we have shown that P is an integrable Kähler polarization on (M, ω). □

We conclude this section by making the following comment. In the symplectic case the Lie subalgebra $(\mathcal{O}_{\mathcal{K}}, \{\cdot,\cdot\})$ is a maximal commutative subalgebra of the algebra of classical observables $(C^\infty(M), \{\cdot,\cdot\})$. The corresponding Hamiltonian vector fields $X_{\lambda_1}, \ldots X_{\lambda_N} \in \Gamma^\infty(P)$ span the *Kähler polarization* P in the sense of Kostant–Souriau geometric quantization.

3. Quantum polarization

Let $\mathcal{D}$ be the vector subspace of the Hilbert space $\mathcal{M}$ generated by finite combinations of the vectors $K_\alpha(m)$, where $\alpha \in I$ and $m \in \Omega_\alpha$. The linear operator $a : \mathcal{D} \to \mathcal{M}$ such that

$$aK_\alpha(m) = \lambda(m)K_\alpha(m) \tag{29}$$

for any $\alpha \in I$ and $m \in \Omega_\alpha$ will be called an *annihilation operator*. On the other hand the operator a^* conjugated to a we shall call a *creation operator*. The eigenvalue function $\lambda : M \to \mathbb{C}$ is well defined on M since $K_\alpha(m) \neq 0$ and the condition (29) does not depend on the choice of gauge.

In general the annihilation operators are not bounded as they are in the case of the Gaussian coherent states map (see Example 16 of Section 4). In this paper we restrict ourselves to the case when the annihilation operators are bounded.

Proposition 4. *The bounded annihilation operators form a commutative unital Banach subalgebra $\overline{\mathcal{P}}_{\mathcal{K}}$ in the algebra $\mathcal{B}(\mathcal{M})$ of all bounded operators in the Hilbert space $\mathcal{M}$.*

Proof. It follows directly from the definition (29) that for any elements $a_1, a_2 \in \overline{\mathcal{P}}_{\mathcal{K}}$ their product $a_1 a_2$ and linear combination $c_1 a_1 + c_2 a_2$ belong to $\overline{\mathcal{P}}_{\mathcal{K}}$. It is also clear that identity operator $\mathbb{I} \in \overline{\mathcal{P}}_{\mathcal{K}}$. We shall show completeness of the subalgebra $\overline{\mathcal{P}}_{\mathcal{K}} \subset \mathcal{B}(\mathcal{M})$. Let $\{a_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence of annihilation operators and let $\{\lambda_n\}_{n\in\mathbb{N}}$ be the corresponding sequence of their eigenfunctions. From the condition (29)

$$|\lambda_k(m) - \lambda_n(m)| = \frac{\|(a_k - a_n)K_\alpha(m)\|}{\|K_\alpha(m)\|} \leqslant \|a_k - a_n\|$$

for all $\alpha \in I$ and $m \in \Omega_\alpha$. Hence the sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ converges pointwise to some function $\lambda : \mathcal{M} \to \mathbb{C}$. Since $a_n \overset{n\to\infty}{\longrightarrow} a$ converges in the operator norm to some bounded operator $a \in \mathcal{B}(\mathcal{M})$ one has

$$\|(a - \lambda(m)\mathbb{I})K_\alpha(m)\| = \lim_{n\to\infty} \|(a_n - \lambda_n(m)\mathbb{I})K_\alpha(m)\| = 0.$$

Consequently λ is the eigenvalue function for a and $a \in \overline{\mathcal{P}}_{\mathcal{K}}$. The annihilation operators $a_1, a_2 \in \overline{\mathcal{P}}_{\mathcal{K}}$ commute on a dense domain $\mathcal{D} \subset \mathcal{M}$ implying the commutativity of the subalgebra $\overline{\mathcal{P}}_{\mathcal{K}}$. □

The eigenvalue function is the covariant symbol

$$\lambda(m) = \frac{\langle K_\alpha(m)|aK_\alpha(m)\rangle}{\langle K_\alpha(m)|K_\alpha(m)\rangle} =: \langle a\rangle(m) \tag{30}$$

of the annihilation operator. It is thus a bounded complex analytic function on the complex manifold M.

We shall describe now the algebra $\mathcal{P}_{\mathcal{K}} := \{a^*, a \in \overline{\mathcal{P}}_{\mathcal{K}}\}$ of creation operators in terms of the Hilbert space $I(\mathcal{M})$. Let $\Lambda : \mathcal{M} \to \mathcal{M}$ be a linear operator defined in the following way: there exists $\lambda \in \mathcal{O}_{\mathcal{K}}$ such that

$$\lambda I(v) = I(\Lambda v)$$

for all $v \in \mathcal{M}$. The operator defined above has the following properties.

Proposition 5. (i) *Any such operator Λ is bounded on $\mathcal{M}$.*
(ii) *The operator Λ^* adjoint to Λ is an annihilation operator with covariant symbol given by the bounded function $\overline{\lambda}$.*

Proof. (i) From the sequence of equalities

$$\begin{aligned}\langle \overline{\lambda(m)}K_\alpha(m)|v\rangle \overline{s}^*_\alpha(m) &= \lambda(m)\langle K_\alpha(m)|v\rangle \overline{s}^*_\alpha \\ &= \langle K_\alpha(m)|\Lambda v\rangle \overline{s}^*_\alpha(m)\end{aligned} \tag{31}$$

where $v \in \mathcal{M}$, $\alpha \in J$ and $m \in \Omega_\alpha$ it follows that $\mathcal{D}$ is the domain of the conjugated operator Λ^*. Since $\mathcal{D}$ is dense in $\mathcal{M}$ the operator Λ admits the closure $\overline{\Lambda} = \Lambda^{**}$, see [11]. We have $\mathcal{M} = D(\Lambda) \subset D(\overline{\Lambda})$ which implies the boundedness of Λ.

(ii) Let us notice that from (31) it follows that

$$\Lambda^* K_\alpha(m) = \overline{\lambda}(m)K_\alpha(m). \tag{32}$$

Thus Λ^* is an annihilation operator with $\overline{\lambda}$ as its covariant symbol. □

From this two propositions and from

$$\|\langle b\rangle\|_\infty = \sup_{m\in M} |\langle b\rangle(m)| \leqslant \|b\| \tag{33}$$

one can deduce the following

Proposition 6. *The mean value map $\langle\cdot\rangle$ defined by (30) gives a continuous isomorphism of $\overline{\mathcal{P}_{\mathcal{K}}}$ with the function Banach algebra $(\mathcal{O}_{\mathcal{K}}, \|\cdot\|_\infty)$.*

Let us assume that for some measure μ one has the resolution of the identity operator

$$\mathbb{I} = \int_M P(m)\mathrm{d}\mu(m), \tag{34}$$

where

$$P(m) := \frac{|K_\alpha(m)\rangle\langle K_\alpha(m)|}{\langle K_\alpha(m)|K_\alpha(m)\rangle} \tag{35}$$

is the operator of orthogonal projection on the coherent state $\mathcal{K}(m)$, $m \in M$. In such cases the scalar product of the functions $\psi = I(v)$ and $\varphi = I(w)$ can be expressed in terms of the integral

$$\begin{aligned}\langle\psi|\varphi\rangle = \langle v|w\rangle &= \int_M \overline{H}^*(\psi, \varphi)\mathrm{d}\mu \\ &= \int_M \frac{\overline{\langle K_\alpha(m)|v\rangle}\langle K_\alpha(m)|w\rangle}{\langle K_\alpha(m)|K_\alpha(m)\rangle}\mathrm{d}\mu(m).\end{aligned} \tag{36}$$

Moreover one has

$$\|\Lambda v\|^2 = \int_M |\lambda|^2 \overline{H}^*(\psi, \psi)\mathrm{d}\mu \leqslant \|\lambda\|^2_\infty \|v\|^2 \tag{37}$$

for $v \in \mathcal{M}$ and thus it follows that

$$\|\Lambda\| \leqslant \|\lambda\|_\infty. \tag{38}$$

Taking into account the inequalities (33) and (38) we obtain

Theorem 7. *If the coherent states map admits the measure μ which has the resolution of identity property (34) then the mean value map $\langle\cdot\rangle$ is an isometric isomorphism of the Banach algebra $(\overline{\mathcal{P}}_{\mathcal{K}}, \|\cdot\|)$ onto the Banach algebra $(\mathcal{O}_{\mathcal{K}}, \|\cdot\|_\infty)$.*

From the theorem above one may draw the conclusion that the necessary condition for the existence of the identity decomposition for the coherent states map $\mathcal{K}$ is the uniformity of the algebra $\overline{\mathcal{P}}_{\mathcal{K}}$, i.e.

$$\|a^2\| = \|a\|^2 \quad \text{for } a \in \overline{\mathcal{P}}_{\mathcal{K}}.$$

We shall show some facts allowing better understanding of the covariant symbols algebra $\mathcal{O}_{\mathcal{K}}$ in the context of the geometric quantization. According to (ii) of Proposition 5 it is easy to notice that Kostant–Souriau pre-quantization

$$C^\infty(M) \ni \lambda \longrightarrow Q_\lambda = i\overline{\nabla}^*_{X_\lambda} + \lambda \tag{39}$$

after restriction to $\lambda \in \mathcal{O}_{\mathcal{K}}$ gives $Q_\lambda I(v) = \lambda I(v)$ for all $v \in \mathcal{M}$. It shows that the restriction of pre-quantization map $Q_{|\mathcal{O}_{\mathcal{K}}} : \mathcal{O}_{\mathcal{K}} \to \overline{\mathcal{P}}_{\mathcal{K}}$ is inverse to the mean value map $\langle\cdot\rangle : \overline{\mathcal{P}}_{\mathcal{K}} \to \mathcal{O}_{\mathcal{K}}$ given by (30). In the above we have identified $\overline{\mathcal{P}}_{\mathcal{K}}$ with its realization in $I(\mathcal{M})$.

In the light of the remarks above it is strongly justified to call the Banach algebra $\overline{\mathcal{P}}_{\mathcal{K}}$ a *quantum Kähler polarization* of the mechanical system defined by Kähler coherent immersion $\mathcal{K} : M \longrightarrow \mathbb{CP}(\mathcal{M})$.

The next section will be dedicated to purely quantum description of the mechanical system in the C^*-algebra approach.

4. Polarized C^*-algebras as quantum Kähler phase spaces

The function algebra $\mathcal{O}_{\mathcal{K}}$ defines the complex analytic coordinates of the classical phase space (M, ω), i.e. for any $m \in M$ there are open neighborhoods $\Omega \ni m_0$ and $z_1, \ldots, z_N \in \mathcal{O}_{\mathcal{K}}$ such that the map $\varphi : \Omega \to \mathbb{C}^N$ defined by $\varphi(m) := (z_1(m), \ldots, z_N(m))$ for $m \in \Omega$ is a holomorphic chart from the complex analytic atlas of M. It is natural to consider the annihilation operators $a_1, \ldots, a_N \in \overline{\mathcal{P}}_{\mathcal{K}}$ as a quantum complex coordinate system as through the defining relation (29) they correspond to $z_1, \ldots, z_N$.

Let us define *Berezin covariant symbol*

$$\langle F\rangle(m) = \frac{\langle K_\alpha(m)|FK_\alpha(m)\rangle}{\langle K_\alpha(m)|K_\alpha(m)\rangle}, \quad m \in M \tag{40}$$

of the operator F (unbounded in general) whose domain $\mathcal{D}$ contains all finite linear combinations of coherent states. Since $\mathcal{K} : M \to \mathbb{CP}(\mathcal{M})$ is a complex analytic map, the Berezin covariant symbol $\langle F \rangle$ is a real analytic function of the coordinates $\bar{z}_1, \ldots, \bar{z}_N, z_1, \ldots, z_N$.

For $n \in \mathbb{N}$ let $F_n(a_1^*, \ldots, a_N^*, a_1, \ldots, a_N) \in A_{\mathcal{K}}$ be normally ordered polynomials of creation and annihilation operators. We say that

$$F_n(a_1^*, \ldots, a_N^*, a_1, \ldots, a_N) \underset{n\to\infty}{\longrightarrow} F =: F(a_1^*, \ldots, a_N^*, a_1, \ldots, a_N) \tag{41}$$

converges in *coherent state weak topology* if

$$\langle F_n(a_1^*, \ldots, a_N^*, a_1, \ldots, a_N)\rangle(m) \underset{n\to\infty}{\longrightarrow} \langle F\rangle(m). \tag{42}$$

Therefore thinking about observables of the system considered, i.e. self-adjoint operators, as weak coherent state limits of normally ordered polynomials of annihilation and creation operators we are justified in making the

Definition 8. The unital operator C^*-algebra $\mathcal{A}_{\mathcal{K}}$ generated by the Banach algebra $\overline{\mathcal{P}}_{\mathcal{K}}$ we call the *quantum Kähler phase space* generated by the coherent state map $\mathcal{K} : M \to \mathbb{CP}(\mathcal{M})$.

Taking into account the properties of $\mathcal{A}_{\mathcal{K}}$ we define the abstract polarized C^*-algebra.

Definition 9. A *polarized C^*-algebra* is a pair $(\mathcal{A}, \overline{\mathcal{P}})$ consisting of a unital C^*-algebra $\mathcal{A}$ and a Banach commutative subalgebra $\overline{\mathcal{P}}$ such that

(i) $\overline{\mathcal{P}}$ generates $\mathcal{A}$.
(ii) $\overline{\mathcal{P}} \cap \mathcal{P} = \mathbb{C}\mathbb{I}$.

It is easy to see that $\mathcal{A}_{\mathcal{K}}$ is a polarized C^*-algebra in the sense of this definition.

Also the notion of a coherent state can be generalized to the case of abstract polarized C^*-algebra $(\mathcal{A}, \overline{\mathcal{P}})$, namely

Definition 10. A *coherent state* ω on a polarized C^*-algebra $(\mathcal{A}, \overline{\mathcal{P}})$ is a positive linear functional of norm equal to 1 satisfying the condition

$$\omega(xa) = \omega(x)\omega(a) \tag{43}$$

for any $x \in \mathcal{A}$ and any $a \in \overline{\mathcal{P}}$.

Let us stress that in the case when $(\mathcal{A}, \overline{\mathcal{P}})$ is defined by the coherent state map $\mathcal{K} : M \to \mathbb{CP}(\mathcal{M})$ then the state

$$\omega_m(x) := Tr(xP(m)), \tag{44}$$

where $m \in M$ and $P(m)$ is given by (35), is coherent in the sense of Definition 10.

Proceeding as in motivating remarks we shall introduce the notion of a norm normal ordering in polarized C^*-algebra $(\mathcal{A}, \overline{\mathcal{P}})$.

Definition 11. A C^*-algebra $\mathcal{A}$ of quantum observables with fixed polarization $\overline{\mathcal{P}}$ admits *norm normal ordering* if and only if the set of elements of the form

$$\sum_{k=1}^{N} b_k^* a_k$$

where $N \in \mathbb{N}$ and $a_1, \ldots, a_N, b_1, \ldots, b_N \in \overline{\mathcal{P}}$ is dense in $\mathcal{A}$ in C^*-algebra norm topology.

The property of the norm normal ordering is consistent with the Kostant–Souriau quantization on the Kähler manifold. Having this property one can calculate explicitly the covariant symbols (mean values on the coherent states) of the quantum observables. In consequence it gives the relation of the latter to their classical counterparts.

Since we assume that $\mathcal{A}$ is unital the coherent states on $(\mathcal{A}, \overline{\mathcal{P}})$ are positive continuous functionals satisfying the condition $\omega(\mathbb{I}) = 1$. The set of all coherent states on $(\mathcal{A}, \overline{\mathcal{P}})$ will be denoted by $\mathcal{C}(\mathcal{A}, \overline{\mathcal{P}})$. The structure of $\mathcal{C}(\mathcal{A}, \overline{\mathcal{P}})$ is investigated and described in the next section of this paper. Some properties of coherent states are however needed now for the description of algebra $\mathcal{A}_{\mathcal{K}}$ defined by the coherent state map $\mathcal{K} : M \to \mathbb{CP}(\mathcal{M})$.

Theorem 12. *Let $\rho \neq 0$ be a positive linear functional on $(\mathcal{A}, \overline{\mathcal{P}})$. Assume that $\rho \leqslant \omega$, where $\omega \in \mathcal{C}(\mathcal{A}, \overline{\mathcal{P}})$ is a coherent state. Then*

(i) *the functional $\frac{1}{\rho(\mathbb{I})}\rho$ is a coherent state and*

$$\frac{1}{\rho(\mathbb{I})}\rho(a) = \omega(a)$$

for $a \in \overline{\mathcal{P}}$.

(ii) *If $(\mathcal{A}, \overline{\mathcal{P}})$ admits norm normal ordering then*

$$\frac{1}{\rho(\mathbb{I})}\rho = \omega,$$

i.e. the coherent state ω is pure.

Proof. (i) Let $\pi_\omega : \mathcal{A} \longrightarrow \operatorname{End}\mathcal{H}_\omega$ be the GNS representation of $\mathcal{A}$ and let $v_\omega \in \mathcal{H}_\omega$ be the cyclic vector of this representation corresponding to ω. Then there exists an operator $T \in \pi_\omega(\mathcal{A})'$, $0 \leqslant T \leqslant 1$, such that

$$\rho(x) = \langle T v_\omega | \pi_\omega(x) T v_\omega \rangle \tag{45}$$

for any $x \in \mathcal{A}$, see [12,13]. From the defining property (43) of the coherent state one gets

$$\langle v_\omega | \pi_\omega(x)(\pi_\omega(a) - \omega(a)) v_\omega \rangle = 0.$$

Since v_ω is cyclic for $\pi_\omega(\mathcal{A})$ we must have

$$\pi_\omega(a) v_\omega = \omega(a) v_\omega \tag{46}$$

for any $a \in \overline{\mathcal{P}}$. From (45) and (46) it follows that

$$\rho(xa) = \rho(x)\omega(a) \tag{47}$$

for an $x \in \mathcal{A}$ and $a \in \overline{\mathcal{P}}$. Taking $x = \mathbb{I}$ in (47) we get $\frac{1}{\rho(\mathbb{I})}\rho(a) = \omega(a)$. Substituting $\omega(a) = \frac{1}{\rho(\mathbb{I})}\rho(a)$ into (47) and dividing both sides of (47) by $\rho(\mathbb{I}) \neq 0$ we find that $\frac{1}{\rho(\mathbb{I})}\rho$ belongs to $\mathcal{C}(\mathcal{A}, \overline{\mathcal{P}})$.

(ii) Since $\frac{1}{\rho(\mathbb{I})}\rho$ is equal to ω on $\overline{\mathcal{P}}$ we have

$$\frac{1}{\rho(\mathbb{I})}\rho\left(\sum_{k=1}^N b_k^* a_k\right) = \sum_{k=1}^N \overline{\frac{1}{\rho(\mathbb{I})}\rho(b_k)}\frac{1}{\rho(\mathbb{I})}\rho(a_k) = \sum_{k=1}^N \overline{\omega(b_k)}\omega(a_k) = \omega\left(\sum_{k=1}^N b_k^* a_k\right).$$

From the existence of the normal ordering on $(\mathcal{A}, \overline{\mathcal{P}})$ and continuity of ρ and ω it follows that $\frac{1}{\rho(\mathbb{I})}\rho = \omega$ on $\mathcal{A}$. $\square$

Let us remark that the norm normal ordering property of the polarized C^*-algebra $\mathcal{A}$ is stronger than the normal ordering in the Heisenberg quantum mechanics or quantum field theory where it is considered in the weak topology sense.

One of the commonly accepted principles of quantum theory is irreducibility of the algebra of quantum observables. For the Heisenberg–Weyl algebra case the irreducible representations are equivalent to the Schrödinger representation due to the von Neumann theorem [14]. In the case of general coherent states map $\mathcal{K} : M \to \mathbb{CP}(\mathcal{M})$ the irreducibility of the corresponding algebra $\mathcal{A}_\mathcal{K}$ of observables depends on the existence of the norm normal ordering.

Theorem 13. *Let $\mathcal{A}_\mathcal{K}$ be the polarized algebra of observables defined by the coherent states map $\mathcal{K} : M \to \mathbb{CP}(\mathcal{M})$. If M is connected and there exists the norm normal ordering on $\mathcal{A}_\mathcal{K}$ then the auto-representation* $\mathrm{id} : \mathcal{A}_\mathcal{K} \longrightarrow \mathcal{B}(\mathcal{M})$ *is irreducible.*

Proof. It was stated in Theorem 12 that the vector coherent state $\mathcal{K}(m_1)$ is a pure one. This implies irreducibility of the representation

$$\pi_{m_1} := \mathrm{id}_{|\mathcal{M}_{m_1}} : \mathcal{A}_\mathcal{K} \longrightarrow \operatorname{End}\mathcal{M}_{m_1}$$

of the algebra $\mathcal{A}_{\mathcal{K}}$ in the Hilbert subspace $\mathcal{M}_{m_1} = \mathcal{A}\mathcal{K}(m_1)$. There are two possibilities: either $\mathcal{K}(m_2) \subset \mathcal{M}_{m_1}$ for any $m_2 \in M$ or there exists $m_2 \in M$ such that $\mathcal{K}(m_2) \not\subseteq \mathcal{M}_{m_1}$. In the second case it follows from irreducibility of the representation

$$\pi_{m_2} := \mathrm{id}_{|\mathcal{M}_{m_2}} : \mathcal{A}_{\mathcal{K}} \longrightarrow \mathrm{End}\,\mathcal{M}_{m_2}$$

that $\mathcal{M}_{m_2} \subset \mathcal{M}_{m_1}^{\perp}$. Applying this procedure step by step one obtains the orthogonal decomposition

$$\mathcal{M} = \bigoplus_{i \in I} \mathcal{M}_{m_i}$$

of the Hilbert space $\mathcal{M}$. From the assumed separability of $\mathcal{M}$ we find that I is at most countable.

Let

$$M_i := \{m \in M : \mathcal{K}(m) \subset \mathcal{M}_{m_i}\}$$

where $i \in I$. If $m \in M_i \cap \Omega_\alpha$ then

$$\langle K_\alpha(m)|K_\alpha(m)\rangle > 0.$$

Since $K_\alpha : \Omega_\alpha \to \mathbb{C}$ is continuous there exists an open neighborhood $m \in \mathcal{O} \subset \Omega_\alpha$ of m such that

$$\langle K_\alpha(m)|K_\alpha(m')\rangle \neq 0$$

for $m' \in \mathcal{O}$. The following inclusion must be valid: $\mathcal{K}(\mathcal{O}) = [K_\alpha(\mathcal{O})] \subset \mathcal{M}_{m_i}$. Otherwise one would have

$$\langle K_\alpha(m)|K_\alpha(m')\rangle = 0$$

which contradicts the definition of the set $\mathcal{O}$. In this way we have shown that $\mathcal{O} \subset M_i$ and M_i is open in M. Thus M is the disjoint union

$$M = \bigcup_{i \in I} M_i$$

of the open sets. Since, by assumption, M is connected it must be $M = M_i$ for some $i \in I$. The above means that $\mathcal{M} = \mathcal{A}_{\mathcal{K}}\mathcal{K}(m)$ for any $m \in M$ and consequently the representation

$$\mathrm{id} : \mathcal{A}_{\mathcal{K}} \longrightarrow \mathcal{B}(\mathcal{M})$$

is irreducible. □

In the general case one can decompose the Hilbert space $\mathcal{M} = \bigoplus_{i=1}^{N} \mathcal{M}_i$, where $N \in \mathbb{N}$ or $N = \infty$, on the invariant $\mathcal{A}_{\mathcal{K}}\mathcal{M}_i \subset \mathcal{H}_i$ orthogonal Hilbert subspaces. Superposing $\mathcal{K} : M \to \mathbb{CP}(\mathcal{M})$ with the orthogonal projectors $P_i : \mathcal{H} \to \mathcal{H}_i$ one obtains the family of coherent state maps $\mathcal{K}_i := P_i \circ \mathcal{K} : M \to \mathbb{CP}(\mathcal{M}_i)$, $i = 1, \ldots, N$. One has $\mathcal{A}_{\mathcal{K}_i} = P_i \mathcal{A}_{\mathcal{K}} P_i$ and the decomposition $\mathcal{A}_{\mathcal{K}} = \bigoplus_{i=1}^{N} A_{\mathcal{K}_i}$ is consistent with the decomposition

$$K_\alpha(m) = \sum_{i=1}^{N} (P_i \circ K_\alpha)(m), \quad m \in \Omega_\alpha \tag{48}$$

of the coherent state map.

Example 14 (*Toeplitz Algebra*). Fix an orthonormal basis $\{|n\rangle\}_{n=1}^{\infty}$ in the Hilbert space M. The coherent states map $\mathcal{K} : \mathbb{D} \to \mathbb{CP}(\mathcal{M})$ is defined by

$$\mathbb{D} \ni z \longrightarrow K(z) := \sum_{n=1}^{\infty} z^n |n\rangle \tag{49}$$

where $\mathcal{K}(z) = [K(z)]$.

Quantum polarization $\overline{\mathcal{P}}_{\mathcal{K}}$ is generated in this case by the one-side shift operator

$$a|n\rangle = |n-1\rangle \tag{50}$$

which satisfies

$$aa^* = \mathbb{I}. \tag{51}$$

From this relation it follows that the algebra $\mathcal{A}_{\mathcal{K}}$ of physical observables generated by the coherent states map (49) is the Toeplitz C^*-algebra. The existence of norm normal ordering in $(\mathcal{A}_{\mathcal{K}}, \overline{\mathcal{P}}_{\mathcal{K}})$ is guaranteed by the property that monomials

$$a^{*k}a^l \quad k, l \in \mathbb{N} \cup \{0\}$$

are linearly dense in $\mathcal{A}_{\mathcal{K}}$.

Let us finally remark that the space $I(\mathcal{M})$ is exactly the Hardy space $H^2(\mathbb{D})$, see [7,8]. According to Theorem 13 the auto-representation of Toeplitz algebra is irreducible as the unit disc $\mathbb{D}$ is connected and there exists the norm normal ordering in $\mathcal{A}_{\mathcal{K}}$.

Example 15. Following [15] one can generalize the construction presented in Example 14 taking

$$\mathbb{D}_{\mathcal{R}} \ni z \longrightarrow K_{\mathcal{R}}(z) := \sum_{n=1}^{\infty} \frac{z^n}{\sqrt{\mathcal{R}(q)\cdots\mathcal{R}(q^n)}}|n\rangle, \tag{52}$$

where $0 < q < 1$ and $\mathcal{R}$ is a meromorphic function on $\mathbb{C}$ such that $\mathcal{R}(q^n) > 0$ for $n \in \mathbb{N} \cup \{\infty\}$ and $\mathcal{R}(1) = 0$. For $z \in \mathbb{D}_{\mathcal{R}} := \{z \in \mathbb{C} : |z| < \sqrt{\mathcal{R}(0)}\}$ one has $K_{\mathcal{R}}(z) \in \mathcal{M}$ and the coherent state map $\mathcal{K}_{\mathcal{R}} : \mathbb{D}_{\mathcal{R}} \to \mathbb{CP}(\mathcal{M})$ is defined by $\mathcal{K}_{\mathcal{R}}(z) = \mathbb{C}K_{\mathcal{R}}(z)$. The annihilation a and creation a^* operators defined by (52) satisfy the relations

$$\begin{aligned}
&a^*a = \mathcal{R}(Q)\\
&aa^* = \mathcal{R}(qQ)\\
&aQ = qQa\\
&Qa^* = qa^*Q,
\end{aligned} \tag{53}$$

where the compact self-adjoint operator Q is defined by $Q|n\rangle = q^n|n\rangle$. Hence one obtains the class of C^*-algebras $\mathcal{A}_{\mathcal{R}}$ parametrized by the meromorphic functions $\mathcal{R}$, which includes the q-Heisenberg–Weyl algebra of one degree of freedom and the quantum disc in the sense of [15] if

$$\mathcal{R}(x) = \frac{1-x}{1-q} \tag{54}$$

and

$$\mathcal{R}(x) = r\frac{1-x}{1-\rho x}, \tag{55}$$

where $0 < r, \rho \in \mathbb{R}$, respectively.

The algebras $\mathcal{A}_{\mathcal{R}}$ find application for the integration of quantum optical models, see [16]. For the rational $\mathcal{R}$ they can also be considered as the symmetry algebras in the theory of the basic hypergeometric series, see [15].

In the paper [17] one investigates the case when $\mathcal{R}$ is an invertible map. Then relations (53) give

$$aa^* = \mathcal{F}(a^*a), \tag{56}$$

where $\mathcal{F} := \mathcal{R} \circ \mathcal{L}_q \circ \mathcal{R}^{-1}$ and $\mathcal{L}_q(x) = qx$. In this case the C^*-algebra $\mathcal{A}_{\mathcal{R}}$ can be considered as the quantum algebra of the dynamical system defined by the function $\mathcal{F}$.

Example 16 (*q-Heisenberg–Weyl Algebra*). Let M be the polydisc $\mathbb{D}_q \times \cdots \times \mathbb{D}_q$, where $\mathbb{D}_q \subset \mathbb{C}$ is the disc of radius $\frac{1}{\sqrt{1-q}}$, $0 < q < 1$. The orthonormal basis in the Hilbert space $\mathcal{M}$ will be parameterized in the following way

$$\{|n_1 \dots n_N\rangle\}$$

where $n_1, \dots, n_N \in \mathbb{N} \cup \{0\}$, and

$$\langle n_1 \dots n_N | k_1 \dots k_N\rangle = \delta_{n_1k_1} \dots \delta_{n_Nk_N}.$$

The coherent states map

$$\mathcal{K} : \mathbb{D}_q \times \cdots \times \mathbb{D}_q \longrightarrow \mathbb{CP}(\mathcal{M})$$

is defined by $\mathcal{K}(z_1, \ldots, z_N) = [K(z_1, \ldots, z_N)]$ where

$$K(z_1, \ldots, z_N) := \sum_{k_1,\ldots,k_N=0}^{\infty} \frac{z_1^{k_1} \cdots z_N^{k_N}}{\sqrt{[k_1]!_q \cdots [k_N]!_q}} |k_1 \ldots k_N\rangle. \tag{57}$$

The standard notation

$$[n] := 1 + \cdots + q^{n-1}$$
$$[n]!_q := [1] \cdots [n]$$

was used in (57).

The quantum polarization $\overline{\mathcal{P}}_{\mathcal{K}}$ is the algebra generated by the operators $a_1, \ldots, a_N$ defined by

$$a_i K(z_1, \ldots, z_N) = z_i K(z_1, \ldots, z_N). \tag{58}$$

It is easy to show that $\|a_i\| = \frac{1}{\sqrt{1-q}}$. Hence $\overline{\mathcal{P}}_{\mathcal{K}}$ is commutative and algebra $\mathcal{A}_{\mathcal{K}}$ of all quantum observables is generated by $\mathbb{I}, a_1, \ldots, a_N, a_1^*, \ldots, a_N^*$ satisfying the relations

$$\begin{aligned}&[a_i, a_j] = [a_i^*, a_j^*] = 0\\ &a_i a_j^* - q a_j^* a_i = \delta_{ij} \mathbb{I}.\end{aligned} \tag{59}$$

The C^*-algebra $\mathcal{A}_{\mathcal{K}}$ is then the q-deformation of the Heisenberg–Weyl algebra, see [18]. The structural relations (59) imply that $a_i^{*k} a_j^l$, where $i, j = 1, \ldots, N$ and $k, l \in \mathbb{N} \cup \{0\}$ form a linearly dense subset in $\mathcal{A}_{\mathcal{K}}$. Consequently $\mathcal{A}_{\mathcal{K}}$ admits the norm normal ordering. Since the polydisc is connected, the auto-representation of $\mathcal{A}_{\mathcal{K}}$ is irreducible.

In the limit $q \to 1$, $\mathcal{A}_{\mathcal{K}}$ becomes the standard Heisenberg–Weyl algebra for which the creation and annihilation operators are unbounded.

Example 17 (*Quantum Complex Minkowski Space*). This example concerns the quantization of the Minkowski space. The construction of the coherent state map for this case and the investigation of C^*-algebra (the quantum complex Minkowski space) obtained by the method presented here are carried out in [19].

5. Coherent states and multiplicative functionals

In this section we will study the properties of the space $\mathcal{C}(\mathcal{A}, \overline{\mathcal{P}})$ of coherent states on the abstract polarized C^*-algebra $(\mathcal{A}, \overline{\mathcal{P}})$ with admissible norm normal ordering. The coherent state $\omega \in \mathcal{C}(\mathcal{A}, \overline{\mathcal{P}})$, after restriction to $\overline{\mathcal{P}}$, becomes the multiplicative linear functional

$$\omega_{|\overline{\mathcal{P}}} : \overline{\mathcal{P}} \longrightarrow \mathbb{C}.$$

Consequently one has the map

$$\rho : \mathcal{C}(\mathcal{A}, \overline{\mathcal{P}}) \longrightarrow M(\overline{\mathcal{P}}) \tag{60}$$

defined by the restriction $\rho(\omega) := \omega_{|\overline{\mathcal{P}}}$, which maps the space of coherent states $\mathcal{C}(\mathcal{A}, \overline{\mathcal{P}})$ into the space of the multiplicative linear functionals on the commutative Banach algebra $\overline{\mathcal{P}}$. From the inequality

$$\|\rho(\omega)\| \leqslant \|\omega\| \tag{61}$$

and from the existence of the normal ordering on $(\mathcal{A}, \overline{\mathcal{P}})$ it follows that ρ is an injective contraction. This allows us to identify $\mathcal{C}(\mathcal{A}, \overline{\mathcal{P}})$ with the subspace $\rho(\mathcal{C}(\mathcal{A}, \overline{\mathcal{P}}))$ of the space $M(\overline{\mathcal{P}})$ of all multiplicative functionals on $\overline{\mathcal{P}}$.

The theorem presented below characterizes multiplicative functionals from the image $\rho(\mathcal{C}(\mathcal{A}, \overline{\mathcal{P}}))$.

Theorem 18. *The multiplicative functional $\mu \in M(\overline{\mathcal{P}})$ can be prolonged to a coherent state on $(\mathcal{A}, \overline{\mathcal{P}})$ iff for any $a_1, \ldots, a_J, b_1, \ldots, b_S \in \overline{\mathcal{P}}$ one has*

$$\left(\sum_{j=1}^{J} a_j^* a_j \leqslant \sum_{s=1}^{S} b_s^* b_s\right) \Rightarrow \left(\sum_{j=1}^{J} |\mu(a_j)|^2 \leqslant \sum_{s=1}^{S} |\mu(b_s)|^2\right). \tag{62}$$

Proof. The elements

$$x = \sum_{j=1}^{J} d_j^* c_j, \tag{63}$$

where $c_i, d_j \in \overline{\mathcal{P}}$, form a *-invariant complex vector subspace V in it. Let multiplicative functional μ satisfy the condition (62). We will define the linear functional $\omega : V \to \mathbb{C}$ by the formula

$$\omega\left(\sum_{j=1}^{J} d_j^* c_j\right) := \sum_{j=1}^{J} \overline{\mu(d_j)} \mu(c_j). \tag{64}$$

In order to show that this definition of ω is correct we notice that

$$d_j^* c_j = \frac{1}{4} \sum_{k=0}^{3} (-i)^k (c_j + i^k d_j)^* (c_j + i^k d_j). \tag{65}$$

Any $x = x^* \in V$ can be thus expressed in the form

$$x = \sum_{j=1}^{J} a_j^* a_j - \sum_{s=1}^{S} b_s^* b_s.$$

If one takes another decomposition

$$x = \sum_{j=1}^{J} a'^*_j a'_j - \sum_{s=1}^{S} b'^*_s b'_s$$

then due to the equality

$$\sum_{j=1}^{J} a_j^* a_j + \sum_{s=1}^{S} b'^*_s b'_s = \sum_{j=1}^{J} a'^*_j a'_j + \sum_{s=1}^{S} b_s^* b_s$$

and from (62) it follows that definition (64) does not depend on the presentation (63) of the Hermitian element $x^* = x \in V$. It is clear that $\omega(x^*) = \overline{\omega(x)}$ for $x \in V$. So, ω is well defined on V. The condition (62) guarantees its positivity. Hence, according to the Lemma 2.10.1 in [12], ω can be prolonged from V to a positive functional on $\mathcal{A}$. Directly from definition (64) it follows that ω satisfies the condition (43) for $x \in V$ and $a \in \overline{\mathcal{P}}$. Since ω is continuous and V is dense in $\mathcal{A}$ the condition (43) is valid for any $x \in \mathcal{A}$. We have proven that μ can be uniquely prolonged to the coherent state $\omega \in \mathcal{C}(\mathcal{A}, \overline{\mathcal{P}})$.

If $\omega \in \mathcal{C}(\mathcal{A}, \overline{\mathcal{P}})$ then due to its positivity and the condition (43) one gets the property (62). $\square$

From the above one may draw the following

Corollary 19. *The image $\rho(\mathcal{C}(\mathcal{A}, \overline{\mathcal{P}}))$ is the compact subset (in the sense of weak topology) in the space $M(\overline{\mathcal{P}})$ of multiplicative linear functionals on the commutative Banach algebra $\overline{\mathcal{P}}$.*

Proof. If $\mu_n \to \mu \in \overline{\mathcal{P}}$ in weak topology, meaning that $\mu_n(a) \to \mu(a)$ for any $a \in M(\overline{\mathcal{P}})$, then μ satisfies the condition (62) provided the μ_n are subject to the same condition. According to Theorem 18 $\mu \in \rho(\mathcal{C}(\mathcal{A}, \overline{\mathcal{P}}))$ if $\mu_n \in \rho(\mathcal{C}(\mathcal{A}, \overline{\mathcal{P}}))$. Hence $\rho(\mathcal{C}(\mathcal{A}, \overline{\mathcal{P}}))$ is a closed subset of the compact set $M(\overline{\mathcal{P}})$ and must be compact too. $\square$

We shall fix some notation related to the GNS construction. Let ω be the state on C^*-algebra $\mathcal{A}$. By N_ω we will denote the left sided ideal in $\mathcal{A}$ containing all elements $x \in \mathcal{A}$ such that $\omega(x^*x) = 0$. By

$$\|[x]\|_\omega^2 = \langle [x]|[x]\rangle_\omega := \omega(x^*x) \tag{66}$$

we shall denote the scalar product of equivalence class

$$[x] := x + N_\omega \in \mathcal{A}/N_\omega = \mathcal{M}_\omega \tag{67}$$

with itself.

The Hilbert space that is the closure of $\mathcal{A}/N_\omega$ in the norm $\|\cdot\|_\omega$ will be denoted by $\mathcal{H}_\omega$. By v_ω we shall denote the element $[\mathbb{I}] \in \mathcal{H}_\omega$. From Theorem 12 there follows

Proposition 20. *Let ω be the coherent state on $(\mathcal{A}, \overline{\mathcal{P}})$. Then:*

(i)

$$\ker\omega = N_\omega + N_\omega^*, \tag{68}$$

where $\ker\omega = \{x \in \mathcal{A} : \omega(x) = 0\}$.

(ii) *The* GNS *representation*

$$\pi_\omega : \mathcal{A} \longrightarrow \operatorname{End}\mathcal{H}_\omega$$

of C^*-algebra $\mathcal{A}$ is irreducible.

We shall now formulate and prove the theorem which gives equivalent criteria for the state to be a coherent one.

Theorem 21. *Let ω be a state on the polarized algebra $(\mathcal{A}, \overline{\mathcal{P}})$. The following conditions:*

(1) $\omega \in \mathcal{C}(\mathcal{A}, \overline{\mathcal{P}})$,
(2) $N_\omega \cap \overline{\mathcal{P}} = \ker\omega \cap \overline{\mathcal{P}}$,
(3) $\pi_\omega(a)v_\omega = \omega(a)v_\omega$ *for* $a \in \overline{\mathcal{P}}$,
(4) $N_\omega \cap \overline{\mathcal{P}}$ *is the maximal ideal in* $\overline{\mathcal{P}}$

are equivalent.

Proof. (1) $\Rightarrow$ (2)

The element $x \in N_\omega$ iff $\pi_\omega(x)v_\omega = 0$. Hence

$$\omega(a) = \langle v_\omega|\pi_\omega(a)v_\omega\rangle = 0$$

for $a \in N_\omega \cap \overline{\mathcal{P}}$. Consequently $N_\omega \cap \overline{\mathcal{P}} \subset \ker\omega \cap \overline{\mathcal{P}}$. Conversely, if $a \in \ker\omega \cap \overline{\mathcal{P}}$ then

$$0 = |\omega(a)|^2 = \omega(a^*a),$$

meaning that $\ker\omega \cap \overline{\mathcal{P}} \subset N_\omega \cap \overline{\mathcal{P}}$.

(2) $\Rightarrow$ (3)

If (2) is satisfied then $a - \omega(a)\mathbb{I} \in N_\omega \cap \overline{\mathcal{P}}$. This in turn implies that

$$\pi_\omega(a - \omega(a)\mathbb{I})v_\omega = 0$$

for any $a \in \overline{\mathcal{P}}$ and consequently the condition (3).

(3) $\Rightarrow$ (1)

In the GNS representation one has

$$\omega(xa) = \langle v_\omega|\pi_\omega(xa)v_\omega\rangle = \langle v_\omega|\pi_\omega(x)\omega(a)v_\omega\rangle = \omega(x)\omega(a)$$

for $x \in \mathcal{A}$ and $a \in \overline{\mathcal{P}}$. The state is thus coherent.

We have shown the equivalence of (1), (2), (3).

(1) $\Rightarrow$ (4)

If $\omega \neq 0$ is a coherent state then $\rho(\omega)$ is a multiplicative functional on $\overline{\mathcal{P}}$. The intersection $\ker\omega \cap \overline{\mathcal{P}}$ is thus the maximal ideal in $\overline{\mathcal{P}}$. Since (1) $\Leftrightarrow$ (2) the set $N_\omega \cap \overline{\mathcal{P}}$ is the maximal ideal in $\overline{\mathcal{P}}$ too.

(4) $\Rightarrow$ (3)

Assume that $N_\omega \cap \overline{\mathcal{P}}$ is the maximal ideal in $\overline{\mathcal{P}}$. One then has

$$\overline{\mathcal{P}} = (N_\omega \cap \overline{\mathcal{P}}) \oplus \mathbb{C}\mathbb{I}$$

and consequently for any $a \in \overline{\mathcal{P}}$ there exists $\alpha \in \mathbb{C}$ such that $a - \alpha\mathbb{I} \in N_\omega \cap \overline{\mathcal{P}}$. This implies

$$\pi_\omega(a - \alpha\mathbb{I})v_\omega = 0. \tag{69}$$

The equality

$$\omega(a) = \langle v_\omega | \pi_\omega(a) v_\omega \rangle = \alpha \tag{70}$$

together with (69) gives (3). This shows (4) $\Rightarrow$ (3). In this way we have shown the equivalence of all conditions of Theorem 21. □

From the equivalence of the conditions (1) and (3) of the theorem above one immediately obtains the following corollary.

Corollary 22. *For any coherent state ω on $(\mathcal{A}, \overline{\mathcal{P}})$ the vector space*

$$\pi_\omega(\mathcal{P})v_\omega = \{\pi_\omega(a)v_\omega : a \in \mathcal{P}\} \tag{71}$$

is dense in the Hilbert space $\mathcal{H}_\omega$ of the GNS *representation.*

The property described above indicates that the formalism developed in this section is a proper generalization of the Fock representation in quantum mechanics and in quantum field theory. This also justifies the interpretation of the commutative algebra $\mathcal{P}$ as the algebra of creation operators.

Let $(\pi_\omega, \mathcal{H}_\omega, v_\omega)$ and $(\pi_\nu, \mathcal{H}_\nu, v_\nu)$ be the pair of irreducible GNS representations corresponding to the coherent states ω and ν. The unitary equivalence of representations $\pi_\omega \sim \pi_\nu$ defines the equivalence relation of the coherent states $\omega \sim \nu$. By $[\omega] \subset \mathcal{C}(\mathcal{A}, \overline{\mathcal{P}})$ we shall denote the equivalence class of coherent states equivalent to ω in the sense above.

Lemma 23. *Let $(\pi_\omega, \mathcal{H}_\omega, v_\omega)$ and $(\pi_\nu, \mathcal{H}_\nu, v_\nu)$ be* GNS *representations generated by the coherent states ω and ν respectively. One then has*

(i) $(\omega \sim \nu) \Leftrightarrow (\exists v \in \mathcal{H}_\omega$ *such that* $\nu(x) = \langle v | \pi_\omega(x) v\rangle$ *for any* $x \in \mathcal{A}) \Leftrightarrow (\exists w \in \mathcal{H}_\nu$ *such that* $\omega(x) = \langle w | \pi_\nu(x) w\rangle$ *for any* $x \in \mathcal{A})$*;*

(ii) *the vector state*

$$\nu(x) = \langle v | \pi_\omega(x) v\rangle, \tag{72}$$

where $v \in \mathcal{H}_\nu$ is a coherent state iff

$$\pi_\omega(a)v = \nu(a)v \tag{73}$$

for any $a \in \overline{\mathcal{P}}$.

Proof. (i) Obvious

(ii) From (72) and the property (43) one obtains

$$\langle \pi_\omega(x^*)v | (\pi_\omega(a) - \nu(a)\mathbb{I})v\rangle = 0. \tag{74}$$

The set of vectors $\pi_\omega(x^*)v$, $x \in \mathcal{A}$, is dense in $\mathcal{H}_\omega$ as the representation π_ω is irreducible. The equality (74) implies then (73). Taking the scalar product of $\pi_\omega(x^*)v$ with (73) one shows the property (43) for ν. □

Lemma 24. (i) *Let $\omega \in \mathcal{C}(\mathcal{A}, \overline{\mathcal{P}})$. Then*

$$\ker \pi_\omega \cap \mathcal{P} = N_\omega \cap \mathcal{P}. \tag{75}$$

(ii) *If $\omega, \nu \in \mathcal{C}(\mathcal{A}, \overline{\mathcal{P}})$ are equivalent then one has the equality*

$$N_\omega \cap \mathcal{P} = N_\nu \cap \mathcal{P} \tag{76}$$

of the corresponding ideals.

Proof. (i) According to the GNS construction the left sided ideals N_ω consist of those $x \in \mathcal{A}$ which annihilate the vector $v_\omega \in \mathcal{H}_\omega : \pi_\omega(x)v_\omega = 0$. This implies

$$\ker \pi_\omega \cap \mathcal{P} \subset N_\omega \cap \mathcal{P}.$$

Since $N_\omega \cap \mathcal{P}$ is an ideal in $\mathcal{P}$ one has

$$\pi_\omega(a)\pi_\omega(b^*)v_\omega = 0$$

for any $a \in N_\omega \cap \mathcal{P}$ and any $b \in \overline{\mathcal{P}}$. The vectors $\pi_\omega(b^*)$, $b \in \overline{\mathcal{P}}$, according to Corollary 22 constitute a dense subspace in $\mathcal{H}_\omega$. One thus has

$$N_\omega \cap \mathcal{P} \subset \ker \pi_\omega \cap \mathcal{P}.$$

(ii) The equivalence $\omega \sim \nu$ of the coherent states ω and ν implies the equality of the kernels

$$\ker \pi_\omega = \ker \pi_\nu$$

of the corresponding representations. From this together with (i) one obtains (76). $\square$

The proposition below gives necessary and sufficient conditions for coherent states to be equivalent.

Proposition 25. *Let ω and ν be coherent states and $(\pi_\omega, \mathcal{H}_\omega, v_\omega)(\pi_\nu, \mathcal{H}_\nu, v_\nu)$ be respectively their corresponding* GNS *representations. Then the following conditions:*

(i) $\omega \sim \nu$,
(ii) $\dim(\pi_\omega(\ker \nu \cap \mathcal{P})v_\omega)^\perp = 1$,
(iii) $\dim(\pi_\nu(\ker \omega \cap \mathcal{P})v_\nu)^\perp = 1$

are equivalent.

Proof. Assume that

$$\dim(\pi_\omega(\ker \nu \cap \mathcal{P})v_\omega)^\perp = 1.$$

Then there exists a vector $v \in (\pi_\omega(\ker \nu \cap \mathcal{P})v_\omega)^\perp$ such that $\|v\| = 1$ and

$$0 = \langle v|\pi_\omega(\ker \nu \cap \mathcal{P})\pi_\omega(\mathcal{P})v_\omega\rangle = \langle \pi_\omega(\ker \nu \cap \overline{\mathcal{P}})v|\pi_\omega(\mathcal{P})v_\omega\rangle. \tag{77}$$

The equality (77) is true due to the fact that $\ker \nu \cap \mathcal{P}$ is an ideal in $\mathcal{P}$. From (77) and because $\pi_\omega(\mathcal{P})v_\omega$ is dense in $\mathcal{H}_\omega$ one has

$$\pi_\omega(a - \nu(a)\mathbb{I})v = 0 \tag{78}$$

for any $a \in \overline{\mathcal{P}}$. One can thus represent the coherent state ν by

$$\nu(x) = \langle v|\pi_\omega(x)v\rangle \tag{79}$$

for $x \in \mathcal{A}$. According to Lemma 23 and (78) and (79) the coherent state ν is equivalent to ω.

Conversely, if one assumes $\nu \sim \omega$, then again due to (ii) of Lemma 23 there exists $v \in \mathcal{H}_\omega$, $\|v\| = 1$ such that

$$\nu(x) = \langle v|\pi_\omega(x)v\rangle$$

for $x \in \mathcal{A}$, and

$$\pi_\omega(a)v = 0 \tag{80}$$

for $a \in \ker \nu \cap \overline{\mathcal{P}}$. Thus one has

$$v \in (\pi_\omega(\ker \nu \cap \mathcal{P})v_\omega)^\perp,$$

showing that $(\pi_\omega(\ker \nu \cap \mathcal{P})v_\omega)^\perp \neq \{0\}$. As the relation $\omega \sim \nu$ is symmetric the conditions $(\pi_\omega(\ker \nu \cap \mathcal{P})v_\omega)^\perp \neq \{0\}$ and $(\pi_\nu(\ker \omega \cap \mathcal{P})v_\nu)^\perp \neq \{0\}$ are equivalent. Since $\pi_\omega(\mathcal{P})v_\omega$ is linearly dense in $\mathcal{H}_\omega$ and the codimension of $\ker \nu \cap \mathcal{P}$ in $\mathcal{P}$ is equal to 1 we have $\dim(\pi_\omega(\ker \nu \cap \mathcal{P})v_\omega)^\perp = 1$. This ends the proof. $\square$

The considerations above show that one can describe the coherent states on $(\mathcal{A}, \overline{\mathcal{P}})$ in terms of multiplicative functionals on the polarization $\overline{\mathcal{P}}$, or equivalently, on the antipolarization $\mathcal{P}$. Let us recall that by definition $\mathcal{P}$ consists of the elements *-conjugated to those of $\overline{\mathcal{P}}$.

Now, we will show a similar possibility on the level of GNS representations $(\pi_\omega, \mathcal{M}_\omega, v_\omega)$ generated by the coherent states $\omega \in C(\mathcal{A}, \overline{\mathcal{P}})$. In order to do this let us consider the quotient vector space $\mathcal{P}/\mathcal{P} \cap N_\omega$. The coherent state ω defines the scalar product

$$\langle [a] | [b] \rangle_\omega := \omega(a^* b) \tag{81}$$

on the vectors $[a], [b] \in \mathcal{P}/\mathcal{P} \cap N_\omega$. The completion of $\mathcal{P}/\mathcal{P} \cap N_\omega$ in the norm $\| \cdot \|_\omega$ given by the scalar product (81) is the Hilbert space H_ω. This space is evidently defined in terms of antipolarization $\mathcal{P}$ and multiplicative functionals $\omega_{|\mathcal{P}}$ given by restriction of ω to $\mathcal{P}$. Since $\mathcal{P} \cap N_\omega$ is the ideal of commutative algebra $\mathcal{P}$ one has the representation $\gamma_\omega : \mathcal{P} \to \mathrm{End}\,(\mathcal{P}/\mathcal{P} \cap N_\omega)$ of $\mathcal{P}$ defined by

$$\gamma_\omega(a)[b] = [ab]. \tag{82}$$

It follows from $\|\gamma_\omega(a)\|_\omega \leqslant \|a\|$ that γ_ω extends to the representation of commutative Banach algebra $\mathcal{P}$ in the Hilbert space H_ω. Taking care of the normal ordering in $(\mathcal{A}, \overline{\mathcal{P}})$ we can extend γ_ω onto the whole C^*-algebra $\mathcal{A}$ by putting

$$\gamma_\omega(x) := \sum_{k=1}^{N} \gamma_\omega(b_k)\gamma_\omega(a_k)^*, \tag{83}$$

where $x = \sum_{k=1}^{N} b_k a_k^*$ and $b_k, a_k \in \mathcal{P}$.

We will show that in the case of $\mathcal{A}$ being the Toeplitz algebra the Hilbert space H_ω is naturally isomorphic to the Hardy space $H^2(\mathbb{D})$. The representation γ_ω is the natural representation of the Toeplitz algebra generated by $H^\infty(\mathbb{D}) = \mathcal{P}$. Motivated by this important example we shall call the representation

$$\gamma_\omega : \mathcal{A} \longrightarrow \mathcal{B}(H_\omega) \tag{84}$$

the *Hardy type representation* of the polarized C^*-algebra $(\mathcal{A}, \overline{\mathcal{P}})$.

The map

$$U_\omega([b]) := \pi_\omega(b) v_\omega, \tag{85}$$

where $b \in \mathcal{P}$, defines an isometry of the unitary space $\mathcal{P}/\mathcal{P} \cap N_\omega$ into the Hilbert space $\mathcal{H}_\omega$. From Corollary 22 it follows that one can extend U_ω to the unitary isomorphism $U_\omega : H_\omega \to \mathcal{H}_\omega$ of the Hilbert spaces since

$$\begin{aligned} U_\omega \gamma_\omega(a)([b]) &= U_\omega([ab]) = \pi_\omega(ab) v_\omega = \pi_\omega(a)(\pi_\omega(b) v_\omega) \\ &= \pi_\omega(a) \circ U_\omega([b]) \end{aligned} \tag{86}$$

for any $b \in \mathcal{P}$. Thus we have

$$U_\omega \circ \gamma_\omega(a) = \pi_\omega(a) \circ U_\omega$$

and

$$U_\omega \circ \gamma_\omega^*(a) = \pi_\omega(a^*) \circ U_\omega. \tag{87}$$

From (83) and (87) we conclude now that U_ω intertwines the Hardy representation with the GNS representation. In order to summarize we formulate

Proposition 26. *The Hardy representation* $(\gamma_\omega, H_\omega)$ *is equivalent to the* GNS *representation* $(\pi_\omega, \mathcal{H}_\omega, v_\omega)$.

In order to stress this property the abbreviated notation HGNS will be used for the representation of Hardy type.

The construction presented above is thus the natural generalization of the construction of the Fock representation for the Heisenberg algebra constructed with creation operators.

6. Classical and quantum Kähler structures

We are now in a position to reconstruct the coherent states map

$$\mathcal{K}_\omega : M_\omega \to \mathbb{CP}(\mathcal{H}_\omega), \tag{88}$$

out of the coherent state ω. The equivalence class $M_\omega := [\omega] \subset M(\overline{\mathcal{P}})$ will be equipped with a weak topology. By $\mathcal{H}_\omega$ we will as usual denote the Hilbert space of the GNS representation $(\pi_\omega, \mathcal{H}_\omega, v_\omega)$.

According to Proposition 25 if $\nu \in M_\omega$ then the dimension of the vector subspace

$$(\pi_\omega(\ker \nu \cap \mathcal{P})v_\omega)^\perp \subset \mathcal{H}_\omega$$

is equal to 1. In addition the vector

$$v \in (\pi_\omega(\ker \nu \cap \mathcal{P})v_\omega)^\perp$$

has property (73) of Lemma 23. Hence, the map $\mathcal{K}_\omega$ given by

$$M_\omega \ni \nu \longrightarrow \mathcal{K}_\omega(\nu) := (\pi_\omega(\ker \nu \cap \mathcal{P})v_\omega)^\perp \in \mathbb{CP}(\mathcal{H}_\omega) \tag{89}$$

satisfies property (29) for the elements $\pi_\omega(a)$ of the commutative operator algebra $\pi_\omega(\overline{\mathcal{P}})$. The common eigenvalues function for the algebra $\pi_\omega(\overline{\mathcal{P}})$ is given by

$$\langle a\rangle(\nu) := \nu(a) = \langle v_\nu|\pi_\omega(a)v_\nu\rangle. \tag{90}$$

It is equal to the restriction $\hat{a}_{|M_\omega}$ of the Gelfand transform image $\hat{a} : M(\overline{\mathcal{P}}) \to \mathbb{C}$ of the element $a \in \overline{\mathcal{P}}$.

The above justifies one to call the map $\mathcal{K}_\omega$ the *coherent states map related* to $\omega \in \mathcal{C}(\mathcal{A}, \overline{\mathcal{P}})$. If one takes the coherent state ω^u equivalent to ω then there exists unitary element $u \in \mathcal{A}$ such that

$$\omega^u(x) = \omega(u^* x u) \tag{91}$$

for any $x \in \mathcal{A}$ (see [12,13]). Then there also exists the unitary isometry $U : \mathcal{H}_\omega \to \mathcal{H}_{\omega^u}$ which is defined by

$$U(\pi_\omega(x)v_\omega) := \pi_{\omega^u}(x)v_{\omega^u} \tag{92}$$

such that

$$\mathcal{K}_{\omega^u} = [U] \circ \mathcal{K}_\omega, \tag{93}$$

where $[U] : \mathbb{CP}(\mathcal{H}_\omega) \to \mathbb{CP}(\mathcal{H}_{\omega^u})$ is the projectivization of U.

In the proof of the theorem below we shall identify all equivalent Hilbert spaces $\mathcal{H}_{\omega^u}$ with $\mathcal{H}_\omega$ and the map U with $\pi_\omega(u)$. The notation of (66) and (67) will be in use.

Theorem 27. *Assume that the* GNS *representation* $(\pi_\omega, \mathcal{H}_\omega, v_\omega)$ *satisfies the condition*

$$\pi_\omega(\mathcal{A}) \cap L^0(\mathcal{H}_\omega) \neq \{0\}, \tag{94}$$

where $L^0(\mathcal{H}_\omega)$ is the ideal of all compact operators in $\mathcal{H}_\omega$. Then the coherent states map $\mathcal{K}_\omega : M_\omega \to \mathbb{CP}(\mathcal{H}_\omega)$ is continuous.

Proof. Let us take the sequence of the coherent states $\omega'_n \in M_\omega$, $n \in \mathbb{N}$, converging to $\omega' \in M_\omega$. Since ω' and ω'_n are pure states there are corresponding unitary elements u and u_n from $\mathcal{A}$ such that

$$\begin{aligned} \omega'(x) &= \omega(u^* x u) \\ \omega'_n(x) &= \omega(u_n^* x u_n) \end{aligned} \tag{95}$$

for any $x \in \mathcal{A}$. One then has

$$\mathcal{K}_\omega(\omega') = [[u]]$$

and

$$\mathcal{K}_\omega(\omega'_n) = [[u_n]] \tag{96}$$

In order to show the convergence

$$[[u_n]] \to [[u]] \tag{97}$$

for $\omega'_n \to \omega'$, we use the equality

$$\begin{aligned}\|[u_n] - [u]\|^2 &= \langle [u_n]_\omega - [u] | [u_n] - [u]\rangle \\ &= \omega(u_n^* u_n) + \omega(u^* u) - \omega(u^* u_n) - \omega(u_n^* u) = 2 - \omega(u^* u_n) - \omega(u_n^* u).\end{aligned} \tag{98}$$

Let us now take the projector $\omega(u^* \cdot)[u] \in L^0(\mathcal{H}_\omega)$. From (94) it follows that $L^0(\mathcal{H}_\omega) \subset \pi_\omega(\mathcal{A})$ (see [13]). So, there is $x_0 \in \mathcal{A}$ such that

$$\pi_\omega(x_0) = \omega(u^* \cdot)[u]. \tag{99}$$

This gives in turn

$$\begin{aligned}\omega'_n(x_0) = \omega(u_n^* x_0 u_n) &= \langle v_\omega | \pi_\omega(u_n^* x_0 u_n) v_\omega\rangle = \langle [u_n] | \omega(u^* \cdot)[u][u_n]\rangle \\ &= \omega(u^* u_n)\langle [u_n] | [u]\rangle = |\omega(u^* u_n)|^2\end{aligned} \tag{100}$$

and

$$\omega'(x_0) = |\omega(u^* u)|^2 = 1.$$

The unitary elements u and u_n can be chosen up to the phase factors $e^{i\varphi}$ and $e^{i\varphi_n}$, where $\varphi, \varphi_n \in \mathbb{R}$. So one can assume $\omega(u^* u_n) \geqslant 0$. From this and from $\omega'_n(x_0) \to \omega'(x_0)$ we conclude that

$$\omega(u^* u_n) \to 1.$$

Hence

$$\|[u_n] - [u]\|^2 \to 0,$$

which implies the convergence of (97). □

If one assumes that it is a postlimital C^*-algebra (see [13]) then the condition (95) is satisfied for any coherent state $\omega \in \mathcal{C}(\mathcal{A}, \overline{\mathcal{P}})$. Consequently for postlimital C^*-algebras the coherent states maps $\mathcal{K}_\omega$ are always continuous.

Now let us consider the commutative Banach algebra homomorphism

$$\Phi_\omega : \mathcal{P} \ni a^* \longrightarrow \overline{\langle a\rangle} \in C(M_\omega), \tag{101}$$

where $C(M_\omega)$ is the algebra of continuous functions on M_ω and $\langle a\rangle$ is defined by (90). According to Lemma 24 one has $\ker \Phi_\omega = \ker \pi_\omega \cap \mathcal{P} = N_\omega \cap \mathcal{P}$. Hence $\ker \Phi_\omega$ depends on $[\omega]$ only and Φ_ω defines the isomorphism of quotient algebra $\mathcal{P}/N_\omega \cap \mathcal{P}$ with function algebra $\mathrm{im}\,\Phi_\omega \subset C(M_\omega)$.

The coherent states map $\mathcal{K}_\omega : M_\omega \to \mathbb{CP}(\mathcal{H}_\omega)$ given by (89) defines line bundle $\mathcal{K}_\omega^*\mathbb{E} =: \mathbb{L} \to M_\omega$ and the anti-linear monomorphism $I : \mathcal{H}_\omega \to \Gamma^\infty(M, \overline{\mathbb{L}^*})$ of complex vector spaces in the way described in Section 2. For $\lambda \in \mathrm{im}\,\Phi_\omega$ and $\psi \in I(\mathcal{H}_\omega)$ one has $\lambda\psi \in I(\mathcal{H}_\omega)$. Thus one is justified in considering $\mathrm{im}\,\Phi_\omega$ as $\mathcal{O}_{\mathcal{K}_\omega}$ and assuming that the function algebra $\mathrm{im}\,\Phi_\omega$ defines the structure of the N-dimensional complex analytic manifold on M_ω. In such a way we come back to the initial data of the constructions presented in Sections 2 and 3, i.e. we reconstruct the classical Kähler phase space from the quantum one.

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